Molecular relaxation effects on sonic boom waveforms: A tutorial survey

Allan D. Pierce
Graduate Program in Acoustics
Pennsylvania State University

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Sonic boom - example of spiked signature

Rise phase of a sonic boom (leading shock in the N-wave)

(This is the early portion of a spiked-signature waveform.)

SR-71 at Mach 2.6; Flight altitude is 66,000 ft

Flying over Mojave desert on August 5, 1987, 5:00 a.m.
What is molecular relaxation?

Nitrogen molecule

Vibrational energy levels

Ground state

First excited state

Number in excited state = function of temperature = function of ground state

But this is only for thermodynamic equilibrium.

Chronology

Taylor (Proc Roy Soc, 1910)
Becker (Zeit Phys, 1922)
Herzfeld and Rice (Phys Rev, 1929)
Knudsen (JASA, 1931, 1933)
Tisza (Phys Rev, 1942)
Lagerstrom, Cole, and Trilling (Cal Tech, 1949)
Markham, Beyer, and Lindsay (Rev Mod Phys, 1951)
Truesdell (J Rat Mech Anal, 1953)
Mendousse (JASA, 1953)
Lighthill (Batchelor, 1956)
Hayes (Emmons, 1958)
Lilley (Lige ICA, 1965)
Clarke and Rodgers (JFM, 1965)
Piercy (JASA, 1969)
Ockendon and Spence (JFM, 1969)
Hodgson and Johannesen (JFM, 1971)
Hodgson (JFM, 1973)
Fowkes-Williams and Howe (JFM, 1973)
Pestorius (UT-ART, 1973)
Anderson (UT-ART, 1974)
Tubb (AIAA, 1975)
Pierce (JSV, 1978)
Bass and Raspet (JASA, 1978)
Johannesen and Hodgson (Rep Prog Phys, 1979)
Morley (AGARD, 1979)
Crighton and Scott (Phil Trans, 1979)
Pierce (Acoustics, 1981)
Holst-Jensen (UTIAS, 1981)
Honma, Glass, Holst-Jensen, and Tsumita (Shock wave symposium, 1981)
Orenstein (UT-ARL, 1982)
Bass, Ezell, and Raspet (JASA, 1983)
Honma and Glass (Proc Roy Soc, 1984)
Lardner and Nicklason (Wave Motion, 1986)
Bass, Layton, Bolen, and Raspet (JASA, 1987)
Kang and Pierce (Anburn, 1990)
Pierce and Kang (ISNA, 1990)
Raspet and Bass (AIAA, 1990)
Kang (PSU, 1991)
Raspet, Bass, Yan, and Wu (NASA 1992)
Parameters characterizing a relaxation process:

- a relaxation time
  \[ \tau_{\text{relax}} \]
- a sound speed increment
  \[ \Delta c = c_{\text{prop}, \text{froz}} - c_{\text{prop}, \text{eq}} \]

Two relaxation processes for air:

- Vibrational relaxation of oxygen molecules
- Vibrational relaxation of nitrogen molecules

\[ \Delta E = \text{Quantum energy gap between ground and first excited vibrational state} \]
\[ k = \text{Boltzmann's constant} \]

When gas is in equilibrium:

Average kinetic energy per molecule (translational plus rotational)
\[ = \frac{5}{2} kT_{\text{eq}} \]

Fraction of molecules in first excited vibrational state
\[ = e^{-\Delta E/kT_{\text{eq}}} \]

For gas not in equilibrium, define apparent temperatures \( T_{\text{tr,rot}} \) and \( T_{\text{vib}} \) such that

Average kinetic energy per molecule (translational plus rotational)
\[ = \frac{5}{2} kT_{\text{tr,rot}} \]

Fraction of molecules in first excited vibrational state
\[ = e^{-\Delta E/kT_{\text{vib}}} \]

Relaxation equation:

\[
\frac{dT_{\text{vib}}}{dt} = \frac{1}{\tau_{\text{relax}}} \left( T_{\text{tr,rot}} - T_{\text{vib}} \right) \\
\left\{ \frac{d}{dt} + \frac{1}{\tau_{\text{relax}}} \right\} \left( T_{\text{vib}} - T_{\text{tr,rot}} \right) = -\frac{d}{dT_{\text{tr,rot}}} T_{\text{tr,rot}}
\]
Relaxation times very sensitive to humidity! Very low humidity means very long relaxation times.

Internal energy per unit mass

$$E_{\text{int}} = E_{\text{tr, rot}} + E_{\text{vib}}$$

Limiting cases for specific heat at constant volume:

- **Equilibrium:**
  $$c_{u,eq} = \frac{dE_{\text{tr,rot}}}{dT_{\text{eq}}} + \frac{dE_{\text{vib}}}{dT_{\text{eq}}}$$

- **Frozen:**
  $$c_{u,frozen} = \frac{dE_{\text{tr,rot}}}{dT_{\text{tr,rot}}}$$

Specific heat ratio:

$$\gamma = \frac{c_p}{c_v} = 1 + \frac{R}{c_u}$$

Sound speed:

$$c_{\text{prop}} = \left[\gamma RT\right]^{1/2}$$

$$c_{\text{prop, froz}} = c_{\text{prop, eq}} + \Delta c$$

$$\Delta c = \text{sound speed increment associated with a relaxation process}$$
Frequency dependence of a single relaxation process

Atmospheric propagation:
Absorption coefficient $\alpha$
(nepers per meter)

Variables:
- $T = 20^\circ$C
- $\text{RH} = 20\%$
- $h = 4.7 \times 10^{-3}$
One-way propagation equation for a dissipative medium
(Burgers' equation)

$$\frac{\partial p}{\partial t} + c \frac{\partial p}{\partial x} + \beta p \frac{\partial p}{\partial x} - \delta \frac{\partial^2 p}{\partial x^2} = 0$$

where

$$2\rho_0 \delta = \frac{4}{3} \mu + \mu_{\text{bulk}} + (\gamma - 1) \frac{\kappa}{c_p}$$

If you neglect nonlinear term, assume $t$ and $x$ dependence as

$$e^{-i\omega t} e^{i(\omega/v_{ph})x} e^{-\alpha x}$$

you get classical absorption coefficient

$$\alpha_{cl} = \frac{\omega^2}{c^3 \delta}$$

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Excerpt (slightly sized) from
The Structure of Shock Waves at Large Distances from Bodies Travelling at High Speeds
G. M. Lilley, 5th ICA, Liege, Belgium, 1965

The Taylor rise time $\tau_{\text{rise}}$ of the shock wave is given as the time over which 0.9 of the overall jump occurs. Thus we find

$$\tau_{\text{rise}} = \frac{6 \rho_0 \delta}{P_{sh}} \left\{ \text{corrected to} \frac{9.8 \rho_0 \delta}{P_{sh}} \right\}$$

where

$$2\rho_0 \delta = \frac{4}{3} \mu + \mu_{\text{bulk}} + (\gamma - 1) \frac{\kappa}{c_p}$$

In estimating diffusivity $\delta$ it is necessary to know the value of the bulk viscosity accurately. Lighthill (Surveys in Mechanics article, 1956) has argued that the presence of minute traces of water vapour in air effects the vibrational energy exchange between the oxygen and water vapour molecules and this results in exceedingly large values of $\mu_{\text{bulk}}$. But for perfectly dry air $\mu_{\text{bulk}} \approx \mu$. (?????) Thus depending on the value of $\mu_{\text{bulk}}$ we find the shock rise time can vary from 4 $\mu$s to 40 $\mu$s for a weak shock wave of 1 lb/ft$^2$ (50 Pa) pressure jump.
Tisza's observation ( paraphrased)

Physical Review, 1942

At sufficiently low frequencies, the effect of any given relaxation process is equivalent to what results from increasing the bulk viscosity by

\[ \Delta \mu_{\text{bulk}} = \left\{ 2 \rho c \right\} \left\{ \tau_{\text{relax}} \right\} \left\{ \Delta c \right\} \]

Perfectly dry air at low frequencies will have a very large bulk viscosity!

Linear dispersion relation with relaxation included

\[ \omega = \left[ c k - ik^2 \delta - k \sum_{\nu} (\Delta c)_{\nu} \frac{ikc \tau_{\nu}}{1 - ikc \tau_{\nu}} \right] \]

How to derive transient wave equation from dispersion relation

\[ -i \omega \to \frac{\partial}{\partial t}; \quad ik \to \frac{\partial}{\partial x} \]

Introduce internal auxiliary variables, \( p_1 \) and \( p_2 \):

\[ \frac{ikc \tau_{\nu}}{1 - ikc \tau_{\nu}} p \to -p_{\nu} \]

\[ c \tau_{\nu} \frac{\partial p}{\partial x} = -p_{\nu} + c \tau_{\nu} \frac{\partial p_{\nu}}{\partial x}; \quad \tau_{\nu} \frac{\partial p}{\partial t} = p_{\nu} + \tau_{\nu} \frac{\partial p_{\nu}}{\partial t} \]

Nonlinear correction (Whitham's rule):

\[ [c] \to c + \nu + \frac{dc}{dp} p = c + \frac{\beta p}{pc} \]
One way nonlinear propagation system
with relaxation included

A generalization of Burgers' equation

\[
\frac{\partial p}{\partial t} + \left[ \frac{c + \beta p}{\rho c} \right] \frac{\partial p}{\partial x} - \frac{\delta^2 p}{\partial x^2} + \sum_{\nu} (\Delta c)_{\nu} \frac{\partial p}{\partial x} = 0
\]

supplemented by relaxation equations (\(\nu = 1, 2\))

\[
ct_{\nu} \frac{\partial p}{\partial x} = -p_{\nu} + ct_{\nu} \frac{\partial p_{\nu}}{\partial x}
\]

or

\[
\tau_{\nu} \frac{\partial p}{\partial t} = p_{\nu} + \tau_{\nu} \frac{\partial p_{\nu}}{\partial t}
\]

Molecular relaxation incorporated
into sonic boom waveform predictions

Burgers' equation with added molecular relaxation term:

\[
\frac{\partial p}{\partial t} + c \frac{\partial p}{\partial x} + NST + TVT + MRT = 0
\]

molecular relaxation term

classical absorption term

nonlinear steepening term

coupled with relaxation equations:

\[
p_{\nu} + \tau_{\nu} \frac{\partial p_{\nu}}{\partial t} = \tau_{\nu} \frac{\partial p}{\partial t}
\]

\(\nu = O_2, N_2\)
Similitude Solution
for waveform in vicinity of shockfront

primary (defendable somewhat) assumption:
\[ p(x, t) = F(x - V_{sh}t); \quad p_{\nu}(x, t) = F_{\nu}(x - V_{sh}t) \]
reduces coupled pde's to coupled ode's

Nominal shock location where \( x - V_{sh}t = 0 \)

To describe shock profile it is sufficient to seek solution corresponding to a net jump:
\[ F(\xi) \to 0 \text{ as } \xi \to \infty \]
\[ F(\xi) \to P_{sh} \text{ as } \xi \to -\infty \]

Complete set of boundary conditions to pin down the solution of the three coupled ode's requires a nontrivial derivation.

Shock speed \( V_{sh} \) emerges as part of the solution.

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Early portion of sonic boom waveform

Theoretical predictions based on numerical solution of augmented Burgers' equation set matched to asymptotic boundary conditions:

Middle rise phase: \( O_2 \) relaxation dominates
Later rise phase: \( N_2 \) relaxation dominates
Early history of shock waveform:
(effects of various terms in the propagation equations)

Definition of rise time as used here
- based on steady state shock overpressure "Psh"
- the time for pressure to jump from 10% to 90% of Psh

Computed results for when
Net pressure jump is 100 Pa
Temperature is 20° C
Relative humidity is 10%

Psh is not always = maximum overpressure
1. All processes included
2. Oxygen relaxation
3. Nitrogen relaxation
4. Viscosity and thermal conduction

HUMIDITY EFFECTS ON SONIC BOOM WAVE FORMS
Rise times of actual sonic boom waveforms recorded at the ground
(Mojave desert, various Mach numbers, flight altitudes, airplanes)
(rel hum \approx 24\%, T \approx 33^\circ C)

Concluding Remarks

- Relaxation theory predicts rise times of right order of magnitude.

- Theoretical predictions of rise times tend to be lower than observed in field data.

- Strong dependence of relaxation theory rise times on humidity.
  Dry air leads to the longest rise times.

- For booms generated by next generation of civilian supersonic aircraft, nitrogen relaxation effects will be much more important than oxygen relaxation effects.

- Rapidity with which waveform profiles adjust to changes in humidity along flight path is topic for further study.